ON THE CONFORMAL REPRESENTATION OF PLANE CURVES PARTICULARLY FOR THE CASES

$p = 4$, 5, and 6

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ON THE CONFORMAL REPRESENTATION OF PLANE CURVES, PARTICULARLY FOR THE CASES $p = 4, 5,$ *and* 6.

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A given non-homogeneous function of x and y

$$F(\overset{n}{x}, \overset{m}{y}) = o$$

of degree n in x and m in y may, we know, be regarded either as a plane curve or as a Riemann's surface. Klein reaches a number of important results here briefly outlined by considerations based on the latter view.

Let

$$F(\overset{n}{x}, \overset{m}{y}) = o$$

be an irreducible algebraic equation defining the surface F_n which is an n-leaved surface spread over the y plane. The deficiency of the surface, p, is fixed by the number of cuts, $2p$, which is necessary to reduce F_n to a simply connected surface. The deficiency so arrived at is numerically the same as the deficiency of the plane curve

$$F(x, y) = o$$

which is precisely the number representing the number of double points which the curve lacks of having the maximum.

An algebraic function or an integral of the first or second kind belongs to a surface when it has but one value for each point on the surface, and when it has only a finite number of infinities and these only algebraic infinities of finite integral order.* Klein proves that upon F_n exist integrals of the first and second kinds. With this work as a basis the surface F_n may be conformally represented by another much simpler sur-

* See Klein Theorie der Elliptischen Modulfunctionen, Vol. I., p. 499.

face. In order to do this we must classify surfaces according to their deficiencies and treat each class separately.

In the case $p = o$ there are no cuts on the surface and no integrals of the first kind. Integrals of the second kind exist on all surfaces. Let us select one of these, w, which has a single algebraic infinity. Since there are no period paths on the surface, w assumes only one value for each point of F_n and since w has the one infinity, and only one, it is a function of "weight" one belonging to F_n where the weight of a function is defined as the number of times which it assumes the value ∞, hence the number of times which it assumes any assigned value, for points on F_n. The function w being of weight one assumes one and only one value corresponding to each point of the n-leaved surface

$$F(x, y) = o.$$

These values, real and complex, may be represented by the points in a plane by the ordinary representation of complex numbers. The given n-leaved surface can then be conformally represented upon a plane by means of the real and complex values assumed by w.

If $p = 1$ two cuts are required to make the surface simply connected. We know that on any surface of deficiency p, there exist p linearly independent integrals of the first kind. Here then there exists only one which we will call u. Let the periods across the cuts be w_1 and w_2. If u has the value u_o at a given point, for all the region around containing no branch point the u will vary continuously, and since u can be nowhere infinite, and since its values may be represented by points of a plane just as any complex number is represented, these points must all be within a parallelogram whose boundaries are determined by the limits of the values of the real and imaginary parts of u as it varies over the surface, never crossing a boundary.

If we seek then to represent our entire surface by means of the integral u which has an infinite number of values we get corresponding to a given point of F_n an infinite number of homologous points in similar parallelograms. We will form the doubly periodic functions

$$P(u \mid w_1, w_2) \text{ and } P'(u \mid w_1, w_2)$$

We know that all doubly periodic functions of u, w_1, and w_2 may be expressed rationally in terms of these two,

$$P(u \mid w_1, w_2) = \frac{1}{u^2} + \Sigma \frac{1}{(u - m_1 w_1 - m_2 w_2)^2} - \frac{1}{(m_1 w_1 + m_2 w_2)^2}$$

$$P'(u \mid w_1, w_2) = -2 \Sigma \frac{1}{(u - m_1 w_1 - m_2 w_2)^3}.$$

These two are everywhere finite except for $u = o$, where the former is infinite of the second order. Hence $P(u \mid w_1, w_2)$ is *a* function of weight two belonging to F_n. By means of it we can represent F_n conformally upon a two leaved Riemann's surface.

In case $p > 1$ we desire to build up a function of weight m which shall belong to the surface, by means of which the surface F_n may be conformally represented upon a simpler surface. Suppose that one such function of weight m exists on the surface and let it be represented by w and its m infinities by $y_1, y_2, y_3, \ldots\ldots y_m$. Let these infinities be of the nature

$$\frac{c_i}{y - y_i}.$$

Let

$$y_{y_1} = \frac{1}{y - y_1} - v_1 j_1 - v_2 j_2 - v_3 j_3 - \ldots\ldots - v_p j_p,$$

where the v's are the periods of w for the cuts a_i and the j's are the normal integrals of the first kind, the periods of j_k for the cuts a_i being all zero except the period for a_k which is unity, and the periods for the cuts b_i being λ_{ki},

$$w - c_1 y_{y_1} - c_2 y_{y_2} - c_3 y_{y_3} - \ldots\ldots - c_m y_{y_m}$$

is everywhere finite, the possible infinities disappearing by subtraction, and since it has periods for the $2p$ cuts, it is an integral of the first kind. Moreover, according to definition, the periods for the cuts a_i are all zeros, and therefore this integral can be put equal to a constant,* and

$$w = c_0 + c_1 y_{y_1} + c_2 y_{y_2} + c_3 y_{y_3} + \ldots\ldots + c_m y_{y_m}.$$

In order then for u to have but one value for each point on the surface, the periods across the cuts b_i must be equal to zero and

II $$c_1 \lambda_{1k} + c_2 \lambda_{2k} + c_3 \lambda_{3k} \ldots\ldots + c_m \lambda_{mk} = o.$$

If $m > p + 1$ the c's can be found and F_n can be conformally represented upon an m leaved Riemann's surface spread over the w plane by means of a function of weight m belonging to F_n.

* See Klein Theorie der Elliptischen Modulfunctionen, Vol. I., p. 524.

Let the p linearly independent integrals of the first kind of F_n be $w_1, w_2, w_3, \dots\dots w_p$ and their derivatives with respect to y be $\phi_1, \phi_2, \phi_3, \dots\dots \phi_p$. It is easy to show that the ϕ's so found are linearly independent and that any $p+1$ can be expressed linearly in terms of the other p. By expanding w in the region of the zero points, and the branch points, and then differentiating to find the value of the ϕ's we deduce the fact that the ϕ's have $2p-2$ variable zeros on F_n and $2n$ zeros at the infinite points of F_n.

This work offers an easy proof of the Riemann-Roch Theorem, for the equations (II) become in terms of the ϕ's

$$\text{III}\begin{cases} c_1\,\phi_1(y_1)+c_2\,\phi_1(y_2)+c_3\,\phi_1(y_3)\dots\dots+c_m\,\phi_1(y_m)=o \\ c_1\,\phi_2(y_1)+c_2\,\phi_2(y_2)+c_3\,\phi_2(y_3\dots\dots+c_m\,\phi_2(y_m)=o \\ \vdots \qquad\qquad \vdots \\ c_1\,\phi_p(y_1)+c_2\,\phi_p(y_2)+c_3\,\phi_p(y_3)\dots\dots+c_m\phi_p(y_m)=o \end{cases}$$

$$\text{since } \lambda_k=-2\pi i\left(\frac{dk_j}{dy}\right)^*_{y=y_o}$$

If τ of these equations are dependent upon the rest it is possible to combine the other $p-\tau$ in such a manner as to get these dependent ones, and indeed to get τ equations which shall be linear in the ϕ's and which vanish in all the points $y_1, y_2, y_3, \dots\dots y_m$, which proves that there are τ linearly independent functions which vanish in all the points $y_1, y_2, y_3, \dots\dots y_m$.

By solving the system (III) we can express $p-\tau$ of the c's in terms of the other $m-p+\tau$. These $m-p+\tau$ variables enable us to fix the totality of algebraic functions belonging to F_n which are of weight m or less. The most general function of weight m belonging to F_n contains in general $m-p+\tau+1$ arbitrary constants.

The Riemann-Roch Theorem so proved would hold only for $p>1$. Klein extends it to the cases $p=o$ and $p=1$. He constructs a function

$$w=c_o+\frac{c_1}{w-w_1}+\frac{c_2}{w-w_2}+\dots\dots+\frac{c_m}{w-w_m}$$

which is evidently a function of weight m belonging to the surface. Since there are no ϕ's, $\tau=o$ and the number of arbitrary

* See Klein Theorie der Elliptischen Modulfunctionen, Vol. I., p. 532.

constants is $m+1$, which is the number which the Riemann-Roch Theorem would give. Similarly for $p=1$ one ϕ function exists and one equation of the set (III) $\tau=o$ and the Riemann-Roch Theorem holds here also.

To take up the language of the analytic geometry, we have selected a complete set of linearly independent functions each of weight m belonging to F_n and we use these as co-ordinates, this fixing some sort of curve. Every point of F_n gives rise to a set of values of these functions, or to a point, hence the whole surface F_n may be conformally represented by the points of some curve. The functions which we select for this purpose are the ϕ's of which there are p linearly independent. Their ratios are functions belonging to the surface for they have only a finite number of infinities and these algebraic, and the ratios have one and only one value corresponding to each point of F_n. For if one of them, say $\frac{\phi_1}{\phi_2}$ assumed the same value for two different points of F_n, we should have a relation existing among the coefficients of ϕ_1 and ϕ_2, but by hypothesis the ϕ's are independent and hence their ratios belong to the surface F_n. Since, as before shown, each ϕ becomes zero in $2p-2$ variable points of F_n, each ratio may become ∞ in $2p-2$ points, and zero in as many more, and hence the functions which are ratios of the ϕ's are of weight $2p-2$.

Although the ϕ's are linearly independent certain relations of higher order exist among them. For the case $p=o$, no ϕ function exists and F_n, as we saw, is representable by the points of a plane singly covered, or if we consider only the real points, by the points of a straight line.

For $p=1$ one ϕ function exists. We found that the simplest function which will represent F_n in this case is a function $P\ (u \mid w_1\, w_2\ )$ of weight two. According to the Riemann-Roch Theorem there are $m-p+\tau+1$ or two homogeneous arbitrary constants in our representative function. Hence, since it is represented on a two-leaved Riemann's surface, in the language of curves the simplest representation of $F\ (x,\ y)\ =o$ is a doubly covered straight line.

For $p=2$ two ϕ curves exist. Our normal curve then is a doubly covered straight line since it exists in space of one dimension and the ratio of the ϕ's is a two valued function. No relation can exist between the ϕ's.

For $p=3$ one quartic relation exists among the ϕ's so that our normal curve is a plain quartic. For we know from the Riemann-Roch Theorem that the most general quartic relation among the ϕ's contains $4\,(2\,p-2)-p+\tau+1=14$ arbitrary constants which are just enough to give the most general quartic relation among the ϕ's since if we write out a quartic relation it will contain fifteen terms and by a selection of the fourteen arbitrary constants the function is completely fixed.

For $p=4$ two relations of higher order exist, one of the second degree and one of the third degree. This may be proved as follows: We write out all the homogeneous functions of the second degree obtained by taking the squares of the different ϕ's and their products taken two at a time. Divide each of these by some homogeneous function of the second degree in the ϕ's. According to Riemann such functions like branched on a surface T can be expressed linearly in terms of $3\,p-2$ of them which make $3\,p-3$ linearly independent ones. There are $\frac{p(p+1)}{2}$ different combinations of the ϕ's mentioned above and these can be expressed in terms of $3\,p-3$ independent ones, so there must exist at least $\frac{p\,(p+1)}{2}-(3p-3)$ or $\frac{(p-2)\,(p-3)}{2}$ quadratic relations among the ϕs. Similarly there are $\frac{p\,(p+1)\,(p+2)}{6}$ combinations of the ϕ's of the third degree. Divide each of these combinations by the same cubic relation among the ϕ's. of these quotients $5\,(p-1)$ are independent of each other.* There must exist then at least $\frac{p\,(p+1)\,(p+2)}{6}-5\,(p-1)$ cubic relations among them. But we know from the preceding that there are at least $\frac{(p-2)\,(p-3)}{2}$ quadratic relations among the ϕ's. Cubic relations among the ϕ's could consist of these quadratic relations multiplied by any one of p linearly independent equations of the first degree among the ϕ's. To get the number of cubic relations which do not break up thus we shall have to

* Jahresbericht der Deutschen Mathematiker Vereinigung, Vol. III, p. 445. Math. Ann. Vol. XII, pp. 268-310.

subtract these $\frac{p(p-2)(p-3)}{2}$ from the number previously arrived at. There are therefore at least

$$\frac{p(p+1)(p+2)}{6}-5(p-1)-\frac{p(p-2)(p-3)}{2} \text{ or}$$

$$\frac{-p^3+9p^2-23p+15}{3}$$

cubic relations among the ϕ's independent of the quadratic relations. This result is in accordance with Weber for the case $p=4$,* and it does not contradict Noether's statement that there are $\frac{p(p+1)(p+2)}{6}-5(p-1)$ since I say that there are at least $\frac{-p^3+9p^2-23p+15}{3}$, my formula always giving a smaller number than his.

If $p=4$ we select two functions belonging to the surface, one of weight twelve, the other of weight eighteen, the former of degree two, the latter of degree three in the ϕ's. For the normal curve representing $F(x, y)=o$ we have then a twisted sextic in space of three dimensions and defined by the intersection of these surfaces of second and third degree respectively in the ϕ's. Conversely, any twisted sextic which is the intersection of such surfaces is the normal curve of some $F(x, y)=o$ of deficiency four. For at some point of the common intersection pass a plane tangent to the quadratic surface. It will cut the quadratic in two straight lines real or imaginary, and the cubic surface in a plane cubic. The lines each meet the cubic twice beside the original point. Projecting the twisted sextic from the original point on a plane we get a quintic with two double points which is a curve of deficiency 4 and hence the proposition is proved.

If the quadric surface is an ellipsoid, by projecting the twisted sextic from the highest point the quintic obtained will have no infinite points. Its double points will be the projection of one real and one imaginary point of the sextic, and hence will look like an ordinary point on the curve. If the quadric be a cone one of the double points of the quintic may be real and if the surface be an hyperboloid or paraboloid both double points of the quintic may be real. If the quadric become a cone, by

* Math. Ann., Vol. XIII, p. 47.

projecting the twisted sextic from the vertex of the cone we get a conic three times repeated since each generator of the cone will cut the sextic in three points. This case is referred to later.

Noether has published some work in Vol. 26 of the Mathematische Annalen in which he actually works out the relations which may exist among the ϕ's for the cases $p = 5$, 6 and 7. Käsböhrer has a dissertation on the case $p = 8$.

If $p = 5$ our function of the second degree in the ϕ's contains fifteen homogeneous linear constants. According to the Riemann-Roch Theorem it should contain only $16 - 5 + 1 = 12$ When we fix these twelve there are left then three more, homogeneous and linear. So we see the three linearly independent quadratic relations among the ϕ's. The normal curve in this case is a twisted curve of eighth degree in space of four dimensions. Weber proves* that if we take any three homogeneous functions of degree two in the ϕ's and eliminate two of the variables we shall get a curve of deficiency five, thus proving that any twisted curve of degree eight formed by the intersection of three quadrics in space of four dimensions, represents a $F(x, y) = o$ of deficiency five.

It is very easy to get some properties of curves of higher order out of the properties of the normal curves. To illustrate this take the case $p = 3$ where the normal curve is, as we know, a quartic with no double points. We will prove that a curve of order n and deficiency three may be regarded as the envelope of sixty-three different quadratic sheaves of curves of order $2(n - 3.)$ Six in each set break up into two curves each of order $n - 3$ which pass through all of the double points of the curve of the *nth* degree and have their other intersection on a curve of degree $2(n - 3)$. In particular a sextic of deficiency three possesses 28 tangent cubics which pass through the double points of the sextic and such that they can be arranged into sixty-three sets of twelve each, such that the points of intersection of corresponding cubics in each set shall lie on a curve of order six having the same double points as the original sextic. For if we transform a given curve of deficiency three and order n by means of a net of adjoint curves of order $n - 3$ we get a quartic of defiency three and to the adjoints correspond straight lines. We know that such a quartic has

* Math. Ann., Vol. XIII, p. 44.

twenty-eight bitangents which may be divided into sixty-three sets of twelve each such that the points of intersection of corresponding pairs lie on a conic. This quartic may be regarded as being the envelope of sixteen different quadratic sheaves of conics — each sheaf containing six conics which break up into two straight lines, forming a Steiner complex. Moreover all of these double points lie on the Jacobian of the net to which all of the sheaves belong, this Jacobian being of order six. The proposition then follows as the result of the correspondence bethe curve and the *nth* degree and the quartic.

Another illustration is here taken from the case $p = 4$. We will prove that there are twenty-seven different pairs of points on a curve of degree n and deficiency four which can be taken in sets of three in forty-five different ways to lie on as many adjoint ϕ functions of the original function.

For in this case the normal curve is the twisted sextic — the intersection of a quadric with a cubic surface. We know that through any straight line on a cubic surface can be passed five planes each of which cuts the surface in two or more lines, s o that each line is intersected by ten others — eight outside a plane containing three of them.

Considering then three lines in a plane and the eight lines which cut each, we have twenty-seven lines in all. Each line intersects the quadric surface in two points thus giving two points of the normal curve. There are, therefore, fifty-four different points on the twisted sextic such that they lie by sixes in forty-five different planes since the twenty-seven lines lie by threes in forty-five different planes. Carried over by transformation to a curve of order n we get the proposition above.

In the work hitherto, we have been considering the curves represented as perfectly general. We will now examine some special cases. In order to do this we take up some θ functions.

In
$$\theta(v_1, v_2, v_3, \ldots v_p) = \sum_{-\infty}^{\infty} e^{\pi i (a n^2 + 2 n v)}$$

let an^2 be a complete quadratic function of the n's and n a complete linear function of the v's of which there are p. We will now put for the a's the period moduli of the normal integrals

* Salmon's Geometry of Three Dimensions, p. 769.

along the cuts a_i of a surface T defined by $F_n = o$ and of deficiency p. We will put for the v's the normal integrals diminished each by a constant e. We shall then have

$$\theta(v_1, v_2, v_3 \ldots . v_\rho) = \theta \sum_1^p h \int_\varepsilon^\varphi du_h - e_h$$

where the φ's and ϵ's are arbitrary points and φ's are variable.

These integrals of the first kind exist in the original surface F_n and therefore this θ function is like branched with the surface. If it does not vanish identically we know from the properties of θ functions that it has p zeros on the surface. If however, this θ function does vanish identically one or more of these zeros become arbitrary.

Suppose that the e's are so chosen that θ is different from zero.

Then
$$e_h \equiv \sum_{i=1}^{i=p} \int_{\varepsilon_i}^{x_i} du_h + k_h,$$

where x_i are zero points of the θ function and k_h is independpendent of e_h. If moreover, the e's are so chosen that

$$\theta\ (e_1,\ e_2, \ldots . e_p\ ) = 0,$$

then
$$\theta\,(v_1,\ v_2, \ldots . v_p) = 0,$$

and for an arbitrary point and we put

$$e_h = \sum_{i=1}^{i=p-1} \int_{\varepsilon_i}^{x_i} du_h + k_h \qquad - e_h \equiv \sum_{i=1}^{i=p-1} \int_{\varepsilon_i}^{x'_i} du_h + k_h$$

where the point systems x_i and x'_i belong to an equation $\phi = o$ besides lying on the surface $F_n = o$. The quotient of the product

$$\theta\left(\sum_1^p h \int_\epsilon^x du_h - e_h\right) \times \theta\left(\sum_1^p h \int_\epsilon^x du_h + e_h\right)$$

and the product

$$\theta\left(\sum_1^p h \int_\epsilon^x du_h - f_h\right) \times \theta\left(\sum_1^p h \int_\epsilon^x du_h + f_h\right))$$

may be put equal to $\frac{\phi_1}{\phi_2}$, ϕ_1 and ϕ_2 each having like branches with the original function. $\frac{\phi_1}{\phi_2}$ is a function of weight $2p-2$ since it is zero in $2p-2$ variable points and ∞ in as many. The ϕ's are then adjoint curves of the original function and are of order $n-3$.

Introducing the θ functions with characteristics we know that there are $2^{p-1}(2^p-1)$ odd theta functions and $2^{p-1}(2^p+1)$ even theta functions. In general only the odd theta functions vanish for the zero values of the arguments,* If we assume now that $e_h \equiv -e_h$ and $f_h \equiv -f_h$ we get the zeros of our function before considered to fall together in pairs and there exist $p-1$ points at which the ϕ's are zero of order two. $\sqrt{\phi}$ is an Abelian function and there are $2^{p-1}(2^p \pm 1)$ of these together, one for each different characteristic. In the case of the odd theta functions there are $2^{p-1}(2^p-1)$ ϕ curves tangent to the original curve.

Let us make the assumption that up to any number m our function

$$\theta\left(\sum_{l}^{p} h \int_{\epsilon_i}^{x_i}{}_{i=1}^{i=m-1} du_h \pm \frac{w_h}{2}\right)$$

vanishes identically for all the points, $x_1, x_2, \ldots\ldots x_{m-1}$ and

$$\theta\left(\sum_{l}^{p} h \int_{\epsilon_i}^{x_i}{}_{i=m}^{i=m-l} du_h \pm \frac{w_h}{2}\right)$$

does not vanish.

According to Riemann's work before referred to, the condition is that

$$\theta\,[w]\,(v_1, v_2, - \ldots.\, v_p)$$

with all its derivatives up to and including the $(m-1)st$ but not all the *mth* derivatives must vanish for the zero value of the arguments.

*Ueber das Verschwinden der Theta Functionen, Riemann's Werke, p. 198.

The functions

$$\theta\left(\sum_1^p{}_h\int_{\varepsilon}^{x} du_h - \sum_{i=1}^{i=m-1}\int_{\varepsilon_i}^{x_i} du_h - \frac{w_h}{2}\right), \quad \text{and}$$

$$\theta\left(\sum_1^p{}_h\int_{\varepsilon_i}^{x} du_h + \sum_{i=1}^{i=m-2}\int_{\varepsilon_i}^{x_i} + \frac{w_h}{2}\right)$$

will not vanish identically for the points x_i, ϵ_i, taken arbitrarily, and therefore each of them will vanish in $p-1$ points beside ϵ and both together in $2p-2$ points on a function ϕ. Now among the zeros of the first functions are $x_1, x_2 \ldots\ldots x_{m-1}$ and let the rest be $y_m, y_{m+1}\ldots\ldots y_{p-1}$ and let the zeros of the second one be $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_{m-1}, \beta_m, \beta_{m+1}, \ldots\ldots \beta_{p-1}$. There exists then a function ϕ with the zeros $x_1, x_2, \ldots. x_{m-1}, y_m, y_{m+1}, —— y_{p-1}, \alpha_1, \alpha_2, \ldots.\alpha_{m-1}, \beta_m \beta_{m+1}, \ldots. \beta_{p-1}$.

Moreover since the zeros must satisfy the congruences

$$(e_1, e_2, \ldots.e_p) \equiv \left[\sum_1^p{}_h\left(\int_{\varepsilon_1}^{x_1} du_1 + \int_{\varepsilon_2}^{x_2} du_2 .. \int_{\varepsilon_h}^{x_h} du_h + k_h\right)\right]$$

$$\left(\frac{w_2}{2}, \frac{w_2}{2} .. \frac{w_p}{2}\right) = \left[\sum_1^p{}_h \sum_{i=1}^{i=m-1}\int_{v_i}^{x_i} du_h + \sum_{i=p-1}^{i=m}\int_{\varepsilon_i}^{y_i} du_h + k_h\right]$$

$$\left(-\frac{w_2}{2}, -\frac{w_2}{2} .. -\frac{w_p}{2}\right) = \left[\sum_1^p{}_h \sum_{i=1}^{i=m-1}\int_{\varepsilon_i}^{\alpha_i} du_h + \sum_{i=p-1}^{i=m}\int_{\varepsilon_i}^{\beta_i} du_h + k_h\right]$$

Combining the last two by substraction we obtain

$$\sum_1^p{}_h \sum_{i=1}^{i=m-1}\int_{x_i}^{\alpha_i} du_h + \sum_{i=p-1}^{i=m}\int_{y_i}^{\beta_i} du_h \equiv 0$$

It follows from Abel's theorem that there exists a rational function ϕ which is infinitely small of the first order at the

points α_i β_i and infinitely great of the first order at the points x_i y_i and otherwise continuous and different from zero. This function is expressible as the quotient of two functions ϕ. But since there is a function ϕ which vanishes in all the $2p-2$ points we have the two functions $\tau\,\phi=\phi_1$ and $\frac{\phi}{\tau}=\phi_2$ the first infinitely small of the second order in the points

$$x_1, x_2, x_3 \ldots\ldots\ldots x_{m-1}, y_{m+1}, \ldots\ldots\ldots y_{p-1},$$

the second infinitely small of the second order in the points.

$$\alpha_1, \alpha_2, \alpha_3, \ldots\ldots\ldots \alpha_{m-1}, \beta_m, \beta_{m+1}, \beta_{m+2}, \ldots\ldots\ldots \beta_{p-1}.$$

These functions are then squares of Abelian functions. Eliminating τ we obtain $\sqrt{\phi_1\,\phi_2} = \phi$ or $\phi_1\,\phi_2 = \phi^2$.

If we choose the points x_i otherwise we obtain an arbitrary number of Abelian functions $\sqrt{\phi_1}$, $\sqrt{\phi_2}$, etc., which have the property that the square root of the product of two of them is again an Abelian function. Since $m-1$ zeros are arbitrary we get m linearly independent ϕ's of this sort.

I wish now to examine some special cases which arise here in the vanishing of the θ functions. For surfaces of deficiency 0, 1 or 2 no such relations can exist. For the case $p=3$ there are three linearly independent ϕ's. Ordinarily as we have seen no relation exists among them of lower degree than the fourth. The normal curve in the general case is a quartic which is fixed when we stipulate that it shall be a function belonging to the surface defined by the original equation, and fix its infinities.

If now an even θ function belonging to the surface vanishes identically we get, as we have seen, the relation

$$\phi_1\,\phi_2-\phi^2=0$$

We may regard this as the formal curve and, for the sake of continuity, say that it is doubly covered, thus our quartic relation degenerates into two identical equations of the second degree and the normal curve is a conic doubly covered. This equation may be put into the form

$$(\lambda^2\phi_1+2\lambda\phi_2+\phi)\,(\mu_2\phi_1+{}^2\mu\phi_2+\phi)-[\lambda\mu\phi_1+(\lambda+\mu)\phi_2+\phi]^2=o$$

thus showing up the tangent lines if we regard the ϕ's as coordinates, In the case then that an even θ function vanishes

identically the normal curve is such that it has an infinite number of tangent ϕ curves. Any curve then of the same deficiency, in case the even theta function vanishes identically, has an infinite number of tangent ϕ curves. In particular the sextic of deficiency 3 can be put into the form

$$(\lambda_2\phi+2\lambda\phi_2+\phi_3)\,(\lambda^2\mu\phi_1+2\mu\phi_2+\phi_3)-(\lambda\mu\phi_1+(\lambda+\mu)\phi_2+\phi_3)^2=o$$

thus showing the sextic as the envelope of a quadratic sheaf of adjoint curves of order three. Conversely if we get any sort of a quadratic relation among the ϕ's in the case $p=3$ an even theta function most vanish identically, for we can construct a system of such Abelian functions by putting the conic in the form $L_1L_2-L_3^2=o$ and Weber has proved* that if such a system can be constructed linearly and homogeneously from m independent Abelian functions then there can be found a characteristic w possessing the property that the function $\theta\ [w]$ together with all its derivatives up to and including those of order $m-1$ must vanish identically.

In case then that one quadratic relation exists among the ϕs for $p=3$ the normal curve is a doubly covered conic and the case is hyperelliptic.

For $p=4$ suppose that one even θ function vanishes identically. We then get the relation

$$\phi_1\phi_2-\phi_3^2=o$$

For a general curve of deficiency four the normal curve is, as we know, a twisted sextic made by the intersection of a quadric surface with a cubic surface. Now with the vanishing of the θ functions the quadric surface becomes a cone and the representation is characterized by the fact that the tangent ϕ curves to the original curve correspond to the planes tangent to the cone, and so to the points of tangency of ϕ curves correspond three points on the generator of a cone. The normal curve is a twisted sextic such that it has a G_3^1 or a singly infinite system of points three in a set, such that each group of three lies in a straight line.

Suppose another quadratic relation to exist among the ϕ s. This means that the quadratic relation as determined to represent the surface F_n contains a variable parameter by

* Weber in Vol. XIII, Math. Ann., pp. 34–38.